Fixed-Point Iteration method to solve f(x)=0

A fixed point for a function is a number at which the value of the function does not change when the function is applied:

Definition

The number x is a fixed point for a given function if

$$g(x) = x$$

to find the roots of $f(P)=0$

we can define function g with a fixed point at p in a number of ways for example

$$g(x) = x - f(x)$$

OR

$$g(x) = x + 3f(x) \quad , \ldots$$

Then if g has a fixed point at p,

Then the function

$$f(x) = x - g(x)$$

has a zero at p.

Theorem

(1) if $g \in C[a,b]$ and

$g(x) \in [a,b]$ for all $x \in [a,b]$,

Then g has at least one fixed point in $[a,b]$

(2) If $g'(x)$ exists on (a,b)

and a postive Constant $K<1$ exist with $|g'(x)| \le K \quad \forall x \in (a,b)$

Then there is exactly one fixed point in $[a,b]$

to Conclude

to solve $f(x)=0$

rewrite the given equation

in form

$x = g(x)$.

$x_1 = g(x_0)$

$x_2 = g(x_1)$.

$\vdots$

The choose of $g(x)$ effect the Convergence of the solution or it Diverge at all

$x_{n+1} = g(x_n)$. $n = 0, 1, \dots$

make sure that

$|g'(x)| < 1 \quad \forall x \in (a,b)$

were $r \in (a,b)$

Example (1)

find the roots of

$f(x) = e^{-x} - x$

Solution

rewrite the equation in fixed form equ-ion

$x = g(x)$

Then

$x = e^{-x} \Rightarrow \boxed{x_{n+1} = e^{-x_n}}$

to find the interval that the solution is lies on it

$f(0) = e^{0} - 0 = 1$

$f(1) = e^{-1} - 1 = -ve$, Then $r \in [0,1]$.

Let $x_0 = 0.5$

$x_1 = e^{-0.5} = 0.6065$

$x_2 = e^{-0.6065} = 0.5452$

$x_3 = e^{-0.5452} = 0.5797$

$x_4 = \quad = 0.5601$

$x_5 = \quad = 0.5712$

$x_6 = \quad = 0.5649$

* Then The root of The above eq is 0.5649 with error value of $f(0.5649) = -0.003$

Example (2)

Setup an iteration process for

$f(x) = x^2 - 3x + 1 = 0$

Note

Exact Solution 2.6180

1	0.3814

Sol

as it was clear one of root is $\in (0,1)$

The eq may be written in form

$$x = g_1(x) = \frac{1}{3}(x^2+1)$$

$$x_{n+1} = \frac{1}{3}(x_n^2+1)$$

let $x_0 = \frac{0+1}{2} = 0.5$

$$x_1 = \frac{1}{3}(x_0^2+1) = 0.4167$$

$$x_2 = \frac{1}{3}(x_1^2+1) = 0.3912$$

$$x_3 = \qquad = 0.3843$$

$$x_4 = \qquad = 0.3826$$

$$x_5 = \qquad = 0.3821$$

$x_6 = \qquad = 0.3820$

Then the solution is $x = 0.382$

Try another arrange

$x^2 = 3x - 1 \qquad \div x$

$$\boxed{x = 3 - \frac{1}{x}}$$

OR

$$x_{n+1} = 3 - \frac{1}{x_n}$$

See what happend.

Example 3

Find the roots of the equation

$$x^3 + x - 1 = 0$$

using Fixed point method.

Sol $r \in [0, 1]$

Try $x_{n+1} = 1 - x_n^3$

See what happen.

The solution is Divergent

because

$$|G'(x)| = 3x^2 > 1 \text{ on } (a,b)$$

So rewrite the equation $f(x)$
to be convergent
lets try.

$$x_{n+1} = (1+x^2)^{-1}$$

$$G(x) = (1+x^2)^{-1}$$

$$G'(x) = -2x(1+x^2)^{-2}$$

Then $|G'(x)| < 1 \quad \forall x \in (0,1)$

So $x_{n+1} = (1+x^2)^{-1}$ is convergent

and $\boxed{x_6 = 0.7011}$

is the root of the given eq.